Notes: Intro to Time Series and Forecasting – Ch3 ARMA Models

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ARMA(p, q) process: definitions

- $\{X_t\}$ is an ARMA(p, q) process if it is **stationary**, and for all t,

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

  where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and the polynomials

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p, \quad \theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$$

  have no common factors

- Equivalent formula using the backward shift operator

$$\phi(B) X_t = \theta(B) Z_t$$

- An ARMA(p, q) process with mean μ: we can study $\{X_t - \mu\}$

$$(X_t - \mu) - \phi_1 (X_{t-1} - \mu) - \cdots - \phi_p (X_{t-p} - \mu) = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

Stationary solution: existence and uniqueness

- A stationary solution exists and is unique if and only if

$$\phi(z) \neq 0, \quad \text{for all complex } z \text{ with } |z| = 1$$

- The unit circle: the region in $z \in \mathbb{C}$ defined by $|z| = 1$
- Stationary solution:

$$X_t = \theta(B)/\phi(B) Z_t = \psi(B) Z_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

Causality: $\phi(z)$ has no zeros inside the unit circle

- An ARMA(p, q) process is causal: if there exist $\psi_0, \psi_1, \ldots$

$$\sum_{j=0}^{\infty} |\psi_j| < \infty, \quad \text{and}$$

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad \text{for all } t$$

- Theorem (equivalent condition of causaility):

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0, \quad \text{for all } |z| \leq 1$$

- Example: ARMA$(1, 1)$ $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$

$$1 - \phi z = 0 \Longrightarrow \text{ only zero } z = 1/\phi$$

So $|z| = 1/|\phi| > 1$, i.e., $|\phi| < 1$ is equivalent of causality

How do we get ψ_j's?

- Letting $\theta_0 = 1$ and matching coefficients of z^j based on

$$1 + \theta_1 z + \cdots \theta_q z^q = (1 - \phi_1 z - \cdots \phi_p z^p)(\psi_0 + \psi_1 z + \cdots),$$

  gives

$$\theta_j \mathbf{1}_{[j \leq q]} = \psi_j - \sum_{j=1}^{p} \phi_k \psi_{j-k}, \quad j = 0, 1, \ldots$$

- Example: causal ARMA$(1,1)$

$$1 = \psi_0$$
$$\theta = \psi_1 - \phi\psi_0 \Longrightarrow \psi_1 = \theta + \psi$$
$$0 = \psi_j - \phi\psi_{j-1} \text{ for } j \geq 2 \Longrightarrow \psi_j = \phi\psi_{j-1}$$

Therefore,

$$\psi_0 = 1, \quad \psi_j = \phi^{j-1}(\theta + \psi) \text{ for } j \geq 1$$

Invertibility: $\theta(z)$ has no zeros inside the unit circle

- An ARMA(p,q) process is invertible: if there exist $\pi_0, \pi_1, \ldots$

$$\sum_{j=0}^{\infty} |\pi_j| < \infty, \quad \text{and}$$

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}, \quad \text{for all } t$$

- Theorem (equivalent condition of invertibility):

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0, \quad \text{for all } |z| \leq 1$$

- Example: ARMA$(1,1)$ $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$

$$1 + \theta z = 0 \Longrightarrow \text{ only zero } z = -1/\theta$$

So $|z| = 1/|\theta| > 1$, i.e., $|\theta| < 1$ is equivalent of invertibility

How do we get π_j's?

- Letting $\phi_0 = -1$ and matching coefficients of z^j based on

$$1 - \phi_1 z - \cdots \phi_p z^p = (1 + \theta_1 z + \cdots \theta_q z^q)(\pi_0 + \pi_1 z + \cdots),$$

  gives

$$-\phi_j \mathbf{1}_{[j \le p]} = \pi_j + \sum_{j=1}^{q} \theta_k \pi_{j-k}, \quad j = 0, 1, \ldots$$

- Example: invertible ARMA$(1,1)$

$$\begin{aligned} 1 &= \pi_0 \\ -\phi &= \pi_1 + \theta\psi_0 \Longrightarrow \pi_1 = -(\psi + \theta) \\ 0 &= \pi_j + \theta\pi_{j-1} \text{ for } j \ge 2 \Longrightarrow \pi_j = -\theta\pi_{j-1} \end{aligned}$$

Therefore,

$$\pi_0 = 1, \quad \pi_j = (-1)^j \theta^{j-1}(\psi + \theta) \text{ for } j \ge 1$$

Calculation of the ACVF

- Assume the ARMA(p, q) process $\{X_t\}$ is causal and invertible
- Method 1: If $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, then

$$\gamma(h) = E(X_{t+h}E_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$$

- Method 2 (difference equation method): multiple the ARMA formula with $X_t, X_{t-1}, \ldots$ and take expectation

Example: ARMA$(1,1)$

- Recall that for a causal ARMA$(1,1)$, in $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$,

$$\psi_0 = 1, \quad \psi_j = \phi^{j-1}(\theta + \psi) \text{ for } j \geq 1$$

- Lag-0 autocorrelation

$$\gamma(0) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 = \sigma^2 \left[1 + (\theta + \phi)^2 \sum_{j=0}^{\infty} \phi^{2j} \right] = \sigma^2 \left[1 + \frac{(\theta + \phi)^2}{1 - \phi^2} \right]$$

- Lag-1 autocorrelation

$$\gamma(1) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+1} = \sigma^2 \left[\theta + \phi + \frac{(\theta + \phi)^2 \phi}{1 - \phi^2} \right]$$

- Lag-k autocorrelation ($k \geq 2$)

$$\gamma(k) = \phi^{k-1} \gamma(1), \quad k \geq 2$$

Use the difference equation method on **ARMA**$(1, 1)$

1. Multiple $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$ by X_t, then take expectation

$$E(X_t^2) - \phi E(X_t X_{t-1}) = E(X_t Z_t) + \theta E(X_t Z_{t-1})$$

Since $E(X_t Z_k) = E[(\sum_{j=0}^{\infty} \psi_j Z_{t-j}) Z_k] = \psi_{t-k}\sigma^2$, we have

$$\gamma(0) - \phi\gamma(1) = \sigma^2 + \theta(\theta + \phi)\sigma^2$$

2. Multiply by X_{t-1}

$$E(X_{t-1} X_t) - \phi E(X_{t-1}^2) = E(X_{t-1} Z_t) + \theta E(X_{t-1} Z_{t-1})$$

$$\gamma(1) - \phi\gamma(0) = 0 + \theta\sigma^2\psi_0 = \theta\sigma^2$$

Using the two equations from 1 and 2, we can solve $\gamma(0), \gamma(1)$

3. Multiply by X_{t-k}, for $k \geq 2$

$$E(X_{t-k} X_t) - \phi E(X_{t-k} X_{t-1}) = E(X_{t-k} Z_t) + \theta E(X_{t-k} Z_{t-1})$$

$$\gamma(k) - \phi\gamma(k-1) = 0 \Longrightarrow \gamma(k) = \phi\gamma(k-1)$$

ACF of an MA(q) process

- Suppose $\{X_t\}$ is an MA(q), then $\rho(h) = 0$ for all $h > q$
- By asymptotic normality

$$\hat{\rho}(q+1) \mathrel{\dot\sim} \mathrm{N}\left(0, \frac{w_{q+1,q+1}}{n}\right)$$

and Bartlett

$$\begin{aligned} w_{q+1,q+1} &= \sum_{k=1}^{\infty} \left[\rho(k+q+1) + \rho(k-q-1) - 2\rho(k+1)\rho(q)\right]^2 \\ &= \sum_{k=1}^{\infty} \rho(k-q-1)^2 \\ &= 1 + 2\sum_{j=1}^{q} \rho(j)^2 \end{aligned}$$

Test for an MA(q): from the ACF

1. Hypotheses

$$H_0 : \{X_t\} \sim \mathrm{MA}(q) \quad \longleftrightarrow \quad H_A : \text{Not } H_0$$

2. Test statistic

$$Z = \frac{\hat{\rho}(q+1) - 0}{\sqrt{\frac{1+2\sum_{j=1}^{q} \hat{\rho}(j)^2}{n}}}$$

3. Reject H_0 if $|Z| \geq z_{\alpha/2}$

- Note: under the null hypothesis, we use the sample ACF plot with bounds $\pm 1.96 \times \sqrt{\frac{1+2\sum_{j=1}^{q} \hat{\rho}(j)^2}{n}}$ to check if $\hat{\rho}(h)$ for all $h \geq q+1$ are inside the bounds. But this may have some multiple testing problems.

Partial autocorrelation function (PACF)

- We define the partial autocorrelation function (PACF) of an ARMA process as the function $\alpha(\cdot)$

$$\alpha(0) = 1, \quad \alpha(h) = \phi_{hh}, \text{ for } h \geq 1$$

Here, ϕ_{hh} is the last entry of

$$\boldsymbol{\phi}_h = \boldsymbol{\Gamma}_h^{-1}\boldsymbol{\gamma}_h, \quad \text{where } \boldsymbol{\Gamma}_h = [\gamma(i-j)]_{i,j=1}^h, \ \boldsymbol{\gamma}_h = [\gamma(1), \ldots, \gamma(h)]'$$

- Sample PACF $\hat{\alpha}(\cdot)$: change all $\gamma(\cdot)$ above to $\hat{\gamma}(\cdot)$
- Recall: in DL algorithm $\hat{X}_{n+1} = \phi_{n1}X_n + \cdots + \phi_{nn}X_1$,

$$\phi_{nn} = \alpha(n), \quad \text{PACF at lag } n$$

PACF property

- ϕ_{nn} is the correlation between the prediction errors

  $$\alpha(n) = \text{Corr}\left(X_n - P(X_n|X_1, \ldots, X_{n-1}), X_0 - P(X_0|X_1, \ldots, X_{n-1})\right)$$

- Theorem: A stationary series is AR(p) if and only if

  $$\alpha(h) = 0 \text{ for all } h > p$$

- If $\{X_t\}$ is an AR(p), then we have asymptotic normality

  $$\hat{\alpha}(h) \dot{\sim} \mathrm{N}\left(0, \frac{1}{n}\right), \quad \text{for all } h > p$$

Test for an AR(p): from the PACF

1. Hypotheses

$$H_0 : \{X_t\} \sim \mathrm{AR}(p) \quad \longleftrightarrow \quad H_A : \text{Not } H_0$$

2. Test statistic

$$Z = \frac{\hat{\alpha}(p+1) - 0}{\sqrt{\frac{1}{n}}}$$

3. Reject H_0 if $|Z| \geq z_{\alpha/2}$

- Note: under the null hypothesis, we use the sample PACF plot with bounds $\pm 1.96/\sqrt{n}$ to check if $\hat{\alpha}(h)$ for all $h \geq p+1$ are inside the bounds. But this may have some multiple testing problems.

Forecast ARMA(p, q) using the innovation algorithm

- Let $m = \max(p, q)$
- One-step prediction

$$\hat{X}_{n+1} = \begin{cases} \sum_{j=1}^{n} \theta_{nj} \left(X_{n+1-j} - \hat{X}_{n+1-j} \right), & n < m \\ \sum_{i=1}^{p} \phi_i X_{n+1-i} + \sum_{j=1}^{q} \theta_{nj} \left(X_{n+1-j} - \hat{X}_{n+1-j} \right), & n \geq m \end{cases}$$

  – Special case: AR(p) process

$$\hat{X}_{n+1} = \sum_{i=1}^{p} \phi_k X_{n+1-i}, \quad n \geq p$$

- h-step prediction: for $n > m$ and all $h \geq 1$,

$$P_n X_{n+h} = \sum_{i=1}^{p} \phi_i P_n X_{n+h-i} + \sum_{j=h}^{q} \theta_{n+h-1,j} \left(X_{n+1-j} - \hat{X}_{n+1-j} \right)$$

Innovation algorithm parameters vs MA parameters

- Innovation algorithm parameters converges to the MA parameters: If $\{X_t\}$ is invertible, then as $n \to \infty$,

$$\theta_{nj} \longrightarrow \theta_j, \quad j = 1, 2, \ldots, q$$

- Prediction MSE converges to σ^2: Let

$$v_n = E(X_{n+1} - \hat{X}_{n+1})^2, \quad \text{and } v_n = r_n \sigma^2$$

If $\{X_t\}$ is invertible, then as $n \to \infty$,

$$r_n \longrightarrow 1$$

References

- Brockwell, Peter J. and Davis, Richard A. (2016), *Introduction to Time Series and Forecasting, Third Edition*. New York: Springer